NEW EXAMPLES OF OSSERMAN METRICS WITH NONDIAGONALIZABLE JACOBI OPERATORS

J. CARLOS DÍAZ-RAMOS   EDUARDO GARCÍA-RÍO   RAMÓN VÁZQUEZ-LORENZO

Abstract. Explicit examples of Osserman 4-manifolds with exactly two distinct eigenvalues of the Jacobi operators, α and $\beta = 4\alpha \neq 0$, are given. The former has multiplicity two and is a double root of the minimal polynomial of the Jacobi operators.

1. Introduction

To a large extent, the geometry of a pseudo-Riemannian manifold (M, g) is the study of the curvature $R \in \otimes^4 T^*M$ which is defined by the Levi-Civita connection ∇. Since the whole curvature tensor is difficult to handle, the investigation usually focus on different objects whose properties allow us to recover the curvature tensor. Different functions like the sectional curvature or natural operators associated to the curvature are typical examples, being the Jacobi operator the most natural and widely investigated (cf. [13]). A pseudo-Riemannian manifold (M, g) is said to be *Osserman* if the eigenvalues of the Jacobi operators are constant on the unit pseudo-sphere bundles $S^{\pm}(TM)$. Any two-point homogeneous space is Osserman and the converse is true in the Riemannian ($\dim M \neq 16$) [7], [19], [20] and Lorentzian [2], [9] settings. However, there exist many nonsymmetric Osserman pseudo-Riemannian metrics in other signatures (cf. [10], [13]). In particular, the 4-dimensional globally Osserman manifolds are classified except in signature $(--++)$ where, besides the results in [4], [12] a description of all $(--++)$-Osserman metrics is not yet complete.

Since the eigenvalue structure does not completely determine a self-adjoint operator in the indefinite setting, a pseudo-Riemannian manifold is called *Jordan-Osserman* if the Jordan normal form of the Jacobi operators is constant on $S^{\pm}(TM)$. Clearly Jordan-Osserman implies Osserman, but the converse is not true even in dimension four, where both conditions become equivalent at the algebraic setting (i.e., an algebraic curvature tensor in dimension 4 is Osserman if and only if it is Jordan-Osserman, but there exist 4-manifolds which are globally Osserman but not globally Jordan-Osserman [10]). The structure of a Jordan-Osserman algebraic curvature tensor strongly depends on the signature (p, q) of the metric tensor. Indeed, it has been shown in [15] that the spacelike Jacobi operators of a spacelike Jordan-Osserman algebraic curvature tensor are necessarily diagonalizable whenever $p < q$, but they can be arbitrarily complicated in the neutral case ($p = q$) [16]. However the fact that all known examples of (pointwise) Osserman metrics have either diagonalizable or nilpotent Jacobi operators (see [1], [10], [13], [17] and the

1991 *Mathematics Subject Classification.* 53C50, 53B30.
Key words and phrases. Jacobi operator, Osserman manifold, Szabó metric, Walker metric.
Supported by project BFM2003-02949 (Spain).

references therein), suggested that this should be true in the general case, which was conjectured by several authors. The purpose of this note is to point out the existence of *Osserman metrics whose Jacobi operators are neither diagonalizable nor nilpotent*, thus showing that the structure of Osserman metrics in indefinite signature is subtler than expected.

2. The examples

Let $M = \mathbb{R}^4$ with usual coordinates (x_1, x_2, x_3, x_4). For any arbitrary real valued function $f(x_4)$, define a metric by

$$\begin{aligned} g \;=\;& dx^1 \otimes dx^3 + dx^3 \otimes dx^1 + dx^2 \otimes dx^4 + dx^4 \otimes dx^2 \\ & +(4kx_1^2 - \tfrac{1}{4k} f(x_4)^2) dx^3 \otimes dx^3 + 4kx_2^2 dx^4 \otimes dx^4 \\ & +(4kx_1x_2 + x_2 f(x_4) - \tfrac{1}{4k} f'(x_4))(dx^3 \otimes dx^4 + dx^4 \otimes dx^3), \end{aligned} \tag{2.1}$$

where k is a nonzero constant. Then the Levi–Civita connection is determined by the Christoffel symbols as follows

$$\begin{aligned} &\Gamma_{13}^1 = -\Gamma_{33}^3 = 4kx_1, \\ &\Gamma_{13}^2 = \Gamma_{14}^1 = -\Gamma_{34}^3 = \tfrac{1}{2}\Gamma_{24}^2 = -\tfrac{1}{2}\Gamma_{44}^4 = 2kx_2, \\ &\Gamma_{23}^2 = \Gamma_{24}^1 = -\Gamma_{34}^4 = \tfrac{1}{2}(4kx_1 + f(x_4)), \\ &\Gamma_{33}^1 = 16k^2x_1^3 - x_1 f(x_4)^2, \\ &\Gamma_{33}^2 = x_1(16k^2x_1x_2 - f'(x_4)) + f(x_4)(4kx_1x_2 + \tfrac{f'(x_4)}{4k}), \\ &\Gamma_{34}^1 = 16k^2x_1^2x_2 + 4kx_1x_2f(x_4) - \tfrac{1}{2}x_1 f'(x_4) - \tfrac{3f(x_4)f'(x_4)}{8k}, \\ &\Gamma_{34}^2 = \tfrac{1}{2}x_2(32k^2x_1x_2 + 8kx_2f(x_4) - f'(x_4)), \\ &\Gamma_{44}^1 = 16k^2x_1x_2^2 + 4kx_2^2 f(x_4) - \tfrac{f''(x_4)}{4k}, \quad \Gamma_{44}^2 = 16k^2x_2^3. \end{aligned} \tag{2.2}$$

A straightforward calculation from (2.2) shows that the curvature tensor, taken with the sign convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, is given by

$$\begin{aligned} R_{1313} &= R_{2424} = -4k, \\ R_{1324} &= R_{1423} = -2k, \\ R_{1334} &= kx_2(4kx_1 + f(x_4)), \\ R_{1434} &= 4k^2x_2^2, \\ R_{2334} &= \tfrac{f(x_4)^2}{4} - 4k^2x_1^2, \\ R_{2434} &= \tfrac{f'(x_4)}{2} - kx_2(4kx_1 + f(x_4)), \\ R_{3434} &= \tfrac{f'(x_4)^2}{4k} + 2kx_1x_2 f'(x_4) - 2kx_2^2 f(x_4)^2 - x_1 f''(x_4) \\ &\quad - f(x_4)\left(8k^2x_1x_2^2 - \tfrac{5}{2}x_2 f'(x_4) - \tfrac{f''(x_4)}{4k}\right). \end{aligned} \tag{2.3}$$

Now, we have

Theorem 2.1. *For any function $f(x_4)$, the metric (2.1) is Osserman of signature $(2,2)$ with eigenvalues $\{0, 4k, k, k\}$. Moreover, the Jacobi operators are diagonalizable if and only if*

$$24kf(x_4)f'(x_4)x_2 - 12kf''(x_4)x_1 + 3f(x_4)f''(x_4) + 4f'(x_4)^2 = 0. \tag{2.4}$$

Otherwise, k is a double root of the minimal polynomial of the Jacobi operators and (M, g) is Jordan-Osserman on the open set where (2.4) does not hold.

Proof. The eigenvalues of the Jacobi operator of an Osserman metric change sign when passing from timelike to spacelike directions. Thus, for the purpose of studying the Osserman property, it is convenient to consider the operator $J_R(X) = g(X, X)^{-1}R_X$ associated to each nonnull vector X, whose eigenvalues must be constant if and only if (M, g) is Osserman. Let $X = \sum_{i=1}^{4} \alpha_i\partial_i$ be a nonnull vector, where $\{\partial_i\}$ denotes the coordinate basis. The associated Jacobi operator $R_X = R(X, \cdot)X$ can be expressed with respect to the coordinate basis $\{\partial_i\}$ as

$$R_X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ -4k\alpha_3^2 & -4k\alpha_3\alpha_4 & a_{33} & a_{34} \\ -4k\alpha_3\alpha_4 & -4k\alpha_4^2 & a_{43} & a_{44} \end{pmatrix} \tag{2.5}$$

with

$$\begin{aligned}
a_{11} &= 5kx_2 f(x_4)\alpha_3\alpha_4 - f(x_4)^2\alpha_3^2 \\
&\quad + 2k(2\alpha_1\alpha_3 + \alpha_2\alpha_4 + 2k(4x_1^2\alpha_3^2 + 5x_1x_2\alpha_3\alpha_4 + x_2^2\alpha_4^2)) - \alpha_3\alpha_4 f'(x_4), \\
a_{12} &= \tfrac{1}{4}\alpha_4(12kx_2 f(x_4)\alpha_4 - 3f(x_4)^2\alpha_3 \\
&\quad + 8k(\alpha_1 + 6kx_1(x_1\alpha_3 + x_2\alpha_4)) - 2\alpha_4 f'(x_4)), \\
a_{13} &= \tfrac{1}{16k}\big(f(x_4)^2\alpha_3(16k\alpha_1 + \alpha_4 f'(x_4)) \\
&\quad + 4f(x_4)\alpha_4(7kx_2\alpha_4 f'(x_4) - 16k^2x_2\alpha_1 + \alpha_4 f''(x_4)) \\
&\quad - 2(4k\alpha_4(2kx_1(x_1\alpha_3 + x_2\alpha_4) - \alpha_1)f'(x_4) \\
&\quad -3\alpha_4^2 f'(x_4)^2 + 8k(4k\alpha_1(\alpha_1 + 4kx_1(x_1\alpha_3 + x_2\alpha_4)) + x_1\alpha_4^2 f''(x_4)))\big), \\
a_{14} &= -\tfrac{1}{16k}\big(f(x_4)^2\alpha_3(\alpha_3 f'(x_4) - 12k\alpha_2) \\
&\quad + 4f(x_4)(4k^2x_2(\alpha_1\alpha_3 + 3\alpha_2\alpha_4) + 7kx_2\alpha_3\alpha_4 f'(x_4) + \alpha_3\alpha_4 f''(x_4)) \\
&\quad + 2(-4k(\alpha_1\alpha_3 + \alpha_2\alpha_4 + 2kx_1\alpha_3(x_1\alpha_3 + x_2\alpha_4))f'(x_4) \\
&\quad + 3\alpha_3\alpha_4 f'(x_4)^2 + 8k(4k(3kx_1\alpha_2(x_1\alpha_3 + x_2\alpha_4) \\
&\quad + \alpha_1(\alpha_2 + kx_2(x_1\alpha_3 + x_2\alpha_4))) - x_1\alpha_3\alpha_4 f''(x_4)))\big), \\
a_{21} &= \alpha_3(3kx_2 f(x_4)\alpha_3 + 2k(\alpha_2 + 6kx_2(x_1\alpha_3 + x_2\alpha_4)) - \alpha_3 f'(x_4)), \\
a_{22} &= 2k\alpha_1\alpha_3 - \tfrac{1}{4}f(x_4)^2\alpha_3^2 + 4k^2x_1^2\alpha_3^2 + 4k\alpha_2\alpha_4 + 5kf(x_4)x_2\alpha_3\alpha_4 \\
&\quad + 20k^2x_1x_2\alpha_3\alpha_4 + 16k^2x_2^2\alpha_4^2 - \tfrac{3}{2}\alpha_3\alpha_4 f'(x_4), \\
a_{23} &= -\tfrac{1}{4k}\big(4k(kx_2\alpha_4(x_1\alpha_3 + x_2\alpha_4) - \alpha_1\alpha_3)f'(x_4) \\
&\quad - kf(x_4)^2\alpha_2\alpha_3 + \alpha_3\alpha_4 f'(x_4)^2 + f(x_4)(4k^2x_2(3\alpha_1\alpha_3 + \alpha_2\alpha_4) \\
&\quad + 9kx_2\alpha_3\alpha_4 f'(x_4) + \alpha_3\alpha_4 f''(x_4)) \\
&\quad + 4k(4k(kx_1\alpha_2(x_1\alpha_3 + x_2\alpha_4) + \alpha_1(\alpha_2 + 3kx_2(x_1\alpha_3 + x_2\alpha_4))) \\
&\quad - x_1\alpha_3\alpha_4 f''(x_4)))\big), \\
a_{24} &= \tfrac{1}{4k}\big(2k\alpha_3(3\alpha_2 + 2kx_2(x_1\alpha_3 + x_2\alpha_4))f'(x_4) + \alpha_3^2 f'(x_4)^2 \\
&\quad + f(x_4)\alpha_3(-16k^2x_2\alpha_2 + 9kx_2\alpha_3 f'(x_4) + \alpha_3 f''(x_4)) \\
&\quad - 4k(4k\alpha_2(\alpha_2 + 4kx_2(x_1\alpha_3 + x_2\alpha_4)) + x_1\alpha_3^2 f''(x_4)))\big),
\end{aligned}$$

$$a_{33} = k(4\alpha_1\alpha_3 + x_2 f(x_4)\alpha_3\alpha_4 + 2\alpha_4(\alpha_2 + 2kx_2(x_1\alpha_3 + x_2\alpha_4))),$$
$$a_{34} = -k\alpha_3(-2\alpha_2 + x_2 f(x_4)\alpha_3 + 4kx_2(x_1\alpha_3 + x_2\alpha_4)),$$

$$a_{43} = \tfrac{1}{4}\alpha_4(f(x_4)^2\alpha_3 - 4kx_2 f(x_4)\alpha_4 + 2(4k(\alpha_1 - 2kx_1(x_1\alpha_3 + x_2\alpha_4)) + \alpha_4 f'(x_4))),$$
$$\begin{aligned} a_{44} = {} & kx_2 f(x_4)\alpha_3\alpha_4 - \tfrac{1}{4}f(x_4)^2\alpha_3^2 \\ & + 2k(\alpha_1\alpha_3 + 2(\alpha_2\alpha_4 + kx_1\alpha_3(x_1\alpha_3 + x_2\alpha_4))) - \tfrac{1}{2}\alpha_3\alpha_4 f'(x_4). \end{aligned}$$

A direct computation of the characteristic polynomial of the Jacobi operators (2.5), using the expressions above, shows that $p_\lambda(J_R(X)) = \lambda(\lambda - 4k)(\lambda - k)^2$, and therefore (2.1) is Osserman with eigenvalues $\{0, 4k, k, k\}$.

Now, in order to analyze the diagonalizability of the Jacobi operators, we consider the minimal polynomials $m_\lambda(J_R(X))$. It follows after some calculations that

$$J_R(X)\cdot(J_R(X) - 4kId)\cdot(J_R(X) - kId) = \frac{k}{4}g(X,X)^{-1}\,\Xi \begin{pmatrix} 0 & 0 & -\alpha_4^2 & \alpha_3\alpha_4 \\ 0 & 0 & \alpha_3\alpha_4 & -\alpha_3^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\Xi = 3f(x_4)(8kx_2 f'(x_4) + f''(x_4)) + 4(f'(x_4)^2 - 3kx_1 f''(x_4)),$$

which shows that (2.4) is the necessary and sufficient condition for diagonalizability of the Jacobi operators. Finally in the open set where Ξ does not vanish (M, g) is Jordan-Osserman and k is a double root of the minimal polynomials $m_\lambda(J_R(X))$.□

Observe that the eigenvalues of the Jacobi operator of a pseudo-Riemannian Osserman metric change sign from spacelike to timelike vectors, and thus they are all zero for null vectors (cf. [10], [13]), which shows that any Osserman metric is null Osserman.

Theorem 2.2. *For any function $f(x_4)$, the metric (2.1) is null Osserman with two-step nilpotent null Jacobi operators.*

Proof. First of all, observe that a vector $U = \sum_{i=1}^4 \alpha_i\partial_i$ is null if and only if

$$\begin{aligned} & 2\alpha_1\alpha_3 + 2\alpha_2\alpha_4 \\ & \quad + \alpha_3^2\left(4kx_1^2 - \tfrac{f(x_4)^2}{4k}\right) + \alpha_3\alpha_4\left(2f(x_4)x_2 + 8kx_1x_2 - \tfrac{f'(x_4)}{2k}\right) + 4kx_2^2\alpha_4^2 = 0. \end{aligned}$$

Now, a tedious but straightforward calculation from (2.5) shows that

$$R_U^2 = g(U,U)\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ -16k^2\alpha_3^2 & -16k^2\alpha_3\alpha_4 & b_{33} & b_{34} \\ -16k^2\alpha_3\alpha_4 & -16k^2\alpha_4^2 & b_{43} & b_{44} \end{pmatrix},$$

where

$$\begin{aligned} b_{11} = {} & k(16k\alpha_1\alpha_3 - 4f(x_4)^2\alpha_3^2 + 17kf(x_4)x_2\alpha_3\alpha_4 \\ & + 2k(\alpha_2\alpha_4 + 2k(x_1\alpha_3 + x_2\alpha_4)(16x_1\alpha_3 + x_2\alpha_4)) - 3\alpha_3\alpha_4 f'(x_4)), \end{aligned}$$
$$b_{12} = \tfrac{k}{4}\alpha_4(56k\alpha_1 - 15(f(x_4) + 4kx_1)((f(x_4) - 4kx_1)\alpha_3 - 4kx_2\alpha_4) - 10\alpha_4 f'(x_4)),$$
$$\begin{aligned} b_{13} = {} & -16k^2\alpha_1^2 + \tfrac{k}{2}\alpha_1(8(f(x_4) + 4kx_1)((f(x_4) - 4kx_1)\alpha_3 - 4kx_2\alpha_4) + 3\alpha_4 f'(x_4)) \\ & + \tfrac{1}{16}\alpha_4(f'(x_4)(5f(x_4)^2\alpha_3 + 44kf(x_4)x_2\alpha_4 - 80k^2x_1(x_1\alpha_3 + x_2\alpha_4) \\ & + 14\alpha_4 f'(x_4)) + 8(f(x_4) - 4kx_1)\alpha_4 f''(x_4)), \end{aligned}$$

$$\begin{aligned}
b_{14}=&\tfrac{1}{16}(-960k^3x_1^2\alpha_2\alpha_3-960k^3x_1x_2\alpha_2\alpha_4+80k^2x_1^2\alpha_3^2f'(x_4)+40k\alpha_2\alpha_4f'(x_4)\\
&+80k^2x_1x_2\alpha_3\alpha_4f'(x_4)-14\alpha_3\alpha_4f'(x_4)^2-5f(x_4)^2\alpha_3(-12k\alpha_2+\alpha_3f'(x_4))\\
&+8k\alpha_1(-32k\alpha_2-2kx_2((f(x_4)+4kx_1)\alpha_3+4kx_2\alpha_4)+3\alpha_3f'(x_4))\\
&+32kx_1\alpha_3\alpha_4f''(x_4)+4f(x_4)\alpha_4(-kx_2(60k\alpha_2+11\alpha_3f'(x_4))-2\alpha_3f''(x_4))),\\
b_{21}=&k\alpha_3(14k\alpha_2+15kx_2((f(x_4)+4kx_1)\alpha_3+4kx_2\alpha_4)-5\alpha_3f'(x_4)),\\
b_{22}=&\tfrac{k}{4}(8k\alpha_1\alpha_3-f(x_4)^2\alpha_3^2+68kf(x_4)x_2\alpha_3\alpha_4\\
&+16k(4\alpha_2\alpha_4+k(x_1\alpha_3+x_2\alpha_4)(x_1\alpha_3+16x_2\alpha_4))-22\alpha_3\alpha_4f'(x_4)),\\
b_{23}=&\tfrac{1}{4}(kf(x_4)^2\alpha_2\alpha_3-16k^3x_1^2\alpha_2\alpha_3-16k^3x_1x_2\alpha_2\alpha_4-4k\alpha_2\alpha_4f'(x_4)\\
&-20k^2x_1x_2\alpha_3\alpha_4f'(x_4)-20k^2x_2^2\alpha_4^2f'(x_4)-\alpha_3\alpha_4f'(x_4)^2+4k\alpha_1(-16k\alpha_2\\
&-15kx_2((f(x_4)+4kx_1)\alpha_3+4kx_2\alpha_4)+5\alpha_3f'(x_4))+8kx_1\alpha_3\alpha_4f''(x_4)\\
&+f(x_4)\alpha_4(-kx_2(4k\alpha_2+21\alpha_3f'(x_4))-2\alpha_3f''(x_4))),\\
b_{24}=&\tfrac{1}{4}(-64k^2\alpha_2^2+2k\alpha_2(-32kx_2((f(x_4)+4kx_1)\alpha_3+4kx_2\alpha_4)+13\alpha_3f'(x_4))\\
&+\alpha_3(f'(x_4)(kx_2(21f(x_4)\alpha_3+20k(x_1\alpha_3+x_2\alpha_4))+\alpha_3f'(x_4))\\
&+2(f(x_4)-4kx_1)\alpha_3f''(x_4))),\\
b_{33}=&k(16k\alpha_1\alpha_3+\alpha_4(2k\alpha_2+kx_2((f(x_4)+4kx_1)\alpha_3+4kx_2\alpha_4)+\alpha_3f'(x_4))),\\
b_{34}=&-k\alpha_3(-14k\alpha_2+kx_2((f(x_4)+4kx_1)\alpha_3+4kx_2\alpha_4)+\alpha_3f'(x_4)),\\
b_{43}=&-\tfrac{k}{4}\alpha_4(-56k\alpha_1-(f(x_4)+4kx_1)((f(x_4)-4kx_1)\alpha_3-4kx_2\alpha_4)-6\alpha_4f'(x_4)),\\
b_{44}=&\tfrac{k}{4}(8k\alpha_1\alpha_3-f(x_4)^2\alpha_3^2+4kf(x_4)x_2\alpha_3\alpha_4\\
&+16k(4\alpha_2\alpha_4+kx_1\alpha_3(x_1\alpha_3+x_2\alpha_4))-6\alpha_3\alpha_4f'(x_4)).
\end{aligned}$$

As U is a null vector we clearly have $R_U^2=0$. Moreover, it follows from (2.5) that if $R_U=0$ then $\alpha_3=\alpha_4=0$ and the Jacobi operator reduces to

$$R_U=-4k\begin{pmatrix}0&0&\alpha_1^2&\alpha_1\alpha_2\\0&0&\alpha_1\alpha_2&\alpha_2^2\\0&0&0&0\\0&0&0&0\end{pmatrix}\tag{2.6}$$

which shows that R_U vanishes if and only if $U=0$. This proves that (M,g) is null Osserman with two-step nilpotent null Jacobi operators. □

Remark 2.3. Note that although the null Jacobi operators are two-step nilpotent, their Jordan normal form is not necessarily constant on the null cone since the corresponding minimal polynomials may admit one or two double roots. For instance, $U=\alpha_1\partial_1+\alpha_2\partial_2$ is a null vector whose associated Jacobi operator is given by (2.6) and hence its Jordan normal form is given by

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}.$$

On the other hand $V=\partial_3$ is a null vector at those points $(0,x_2,x_3,0)$ for any function $f(x_4)$ with $f(0)=0$. Moreover, in such a case the associated Jacobi operator satisfies

$$(R_V)_{(0,x_2,x_3,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -f'(0) & 0 & 0 & \frac{f'(0)^2}{4k} \\ -4k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and thus its Jordan normal form is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

whenever $f'(0) \neq 0$. Hence the null Osserman and the null Jordan-Osserman conditions are not equivalent at the algebraic level for $(--++)$-metrics, in opposition to the nonnull Osserman conditions. The above example shows that, although the algebraic Osserman condition implies the null Osserman condition, there exist Jordan-Osserman algebraic curvature tensors which are not null Jordan-Osserman.

A pseudo-Riemannian manifold is said to be *Szabó* if the covariant derivative of the Jacobi operators $(\nabla_X R)(X,\cdot)X$ has constant eigenvalues on $S^{\pm}(TM)$ [17]. Any Szabó manifold is locally symmetric in the Riemannian [21] and the Lorentzian [18] setting but the higher signature case supports examples with nilpotent Szabó operators (cf. [17] and the references therein). Next we will show that there exist four-dimensional Szabó metrics where the degree of nilpotency of the associated Szabó operators changes at each point depending on the direction, and thus the Szabó and the Jordan-Szabó algebraic conditions are not equivalent in dimension four, in opposition to the Jacobi operator.

Theorem 2.4. *For any function $f(x_4)$, the metric (2.1) is Szabó of signature $(2,2)$ with zero eigenvalues. Moreover, the minimal polynomial of the Szabó operators $(\nabla_X R)(X,\cdot)X$ depends on the direction X at each point and thus metrics (2.1) are not pointwise Jordan-Szabó.*

Proof. Let $X = \sum_{i=1}^4 \alpha_i \partial_i$ be a nonnull vector as in the proof of Theorem 2.1. Then the associated Szabó operator, when expressed in the coordinate basis takes the form

$$\nabla_X R_X = \begin{pmatrix} A & B \\ 0 & {}^tA \end{pmatrix}, \qquad A = \Psi \begin{pmatrix} \alpha_3\alpha_4 & \alpha_4^2 \\ -\alpha_3^2 & -\alpha_3\alpha_4 \end{pmatrix}, \tag{2.7}$$

where $\Psi = 2\alpha_3 f(x_4)f'(x_4) + \alpha_4 f''(x_4)$. Hence the characteristic polynomial of the Szabó operators is $p_\lambda(\nabla_X R_X) = \lambda^4$ (independently of the 2×2-matrix B).

Since the degree of nilpotency depends on B, in order to show that the Szabó and Jordan-Szabó algebraic conditions are not equivalent, we make the special choice $f(x_4) = x_4$. Now, if X and Y are the unit vectors in the direction of $\partial_1 + \partial_3$ and $\partial_2 + \partial_4$, respectively, one has

$$\nabla_X R_X = \begin{pmatrix} 0 & 0 & 0 & 2(\varepsilon_X - 1)x_4 \\ -2x_4 & 0 & 2x_4 & 4(x_1 - \frac{x_4}{8k} + x_2x_4(8kx_1 + x_4)) \\ 0 & 0 & 0 & -2x_4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\nabla_Y R_Y = \begin{pmatrix} 0 & 0 & 6x_2 + 2(3\varepsilon_Y - 5)x_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\varepsilon_Z = g(Z, Z) = \pm 1$. This shows that $\nabla_X R_X$ is three-step nilpotent at most points while $\nabla_Y R_Y$ is two-step nilpotent. $\square$

Remark 2.5. Let $Gr_2^+(M)$ be the Grassmanian of oriented 2-planes in the tangent bundle. For any nondegenerate 2-plane π the *skew-symmetric curvature operator*

$$R(\pi) = \left| \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2 \right|^{-1/2} R(x, y)$$

is a skew-adjoint operator which is independent of the oriented basis $\{x, y\}$ of π. (M, g) is said to be *Ivanov Petrova* (IP for short) if the eigenvalues of $R(\pi)$ are constant on $Gr_2^+(M)$ (see [13] and the references therein for more information on IP manifolds). Now, an easy calculation shows that metrics (2.1) are not IP in general. For instance, $\pi = \{\partial_3, \partial_1\}$ is a nondegenerate plane whose skew-symmetric operator satisfies

$$R(\pi) = \begin{pmatrix} 4k & 0 & 16k^2x_1^2 - f(x_4)^2 & 3kx_2(4kx_1 + f(x_4)) - \frac{1}{2}f'(x_4) \\ 0 & 2k & 3kx_2(4kx_1 + f(x_4)) - f'(x_4) & 8k^2x_2^2 \\ 0 & 0 & -4k & 0 \\ 0 & 0 & 0 & -2k \end{pmatrix}$$

and hence, it has constant eigenvalues $\{2k, 4k, -2k, -4k\}$ independently of the function $f(x_4)$. On the other hand for any function $f(x_4)$ with $f'(0) \neq 0$, it follows that $\pi = \{\partial_3, \partial_4\}$ is a nondegenerate plane at the origin, whose skew-symmetric operator satisfies

$$R(\pi) = \left| \frac{k}{f'(0)} \right| \begin{pmatrix} 0 & f(0)^2 & -\frac{f(0)^2 f'(0)}{4k} & -\frac{3f'(0)^2 + 2f(0)f''(0)}{2k} \\ 0 & 2f'(0) & \frac{f'(0)^2 + f(0)f''(0)}{k} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -f(0)^2 & -2f'(0) \end{pmatrix}$$

which has eigenvalues $\{0, 0, 2k, -2k\}$. This shows that metrics (2.1) are not IP at the origin on planes of signature $(-+)$ for any function $f(x_4)$ with $f'(0) \neq 0$.

Note that for any metric (2.1) the eigenspace corresponding to the double eigenvalue k is of Lorentzian signature (see Remark 3.4), and thus the curvature tensor at each point is completely determined by the diagonalizability of the Jacobi operator, independently of the function $f(x_4)$. In fact, at any point where the Jacobi operators diagonalize (resp., are not diagonalizable) there exist orthonormal bases where the (algebraic) curvature tensor is expressed in terms of the eigenvalues of the Jacobi operators, independently of the function $f(x_4)$ (see [4], [10, Thm. 4.2.2]).

Next, observe that it is possible to give functions $f(x_4)$ satisfying $f'(0) \neq 0$ and $3f(0)f''(0) + 4f'(0)^2 = 0$ (see (2.4)) and therefore the Jacobi operators are diagonalizable at the origin. Also, there exist functions with $f'(0) \neq 0$ and $3f(0)f''(0) + 4f'(0)^2 \neq 0$ and hence with nondiagonalizable Jacobi operators at the origin. Now, it follows from the eigenvalue structure of the skew-symmetric curvature operators corresponding to the planes discussed above, that none of the corresponding (algebraic) curvature tensors may be IP, thus showing that metrics (2.1) are not IP at any point.

3. Some observations

Remark 3.1. It was shown in [12] that the Jacobi operators of a locally symmetric four-dimensional Osserman metric are either diagonalizable or two-step nilpotent. Therefore, no metric (2.1) may be locally symmetric unless their Jacobi operators diagonalize. Indeed, it follows after some calculations that the covariant derivative of the curvature tensor vanishes at a point $(x_1, \dots, x_4)$ if and only if

(i) $f''(x_4) = 0$, (ii) $f(x_4)f'(x_4) = 0$,

(iii) $f'(x_4)^2 x_1 = 0$, (iv) $24kx_2 f'(x_4)^2 + f'''(x_4)(f(x_4) - 4kx_1) = 0$.

Hence $(\mathbb{R}^4, g)$ is locally symmetric if and only if the function f is constant, and thus the Jacobi operators are diagonalizable from (2.4). Furthermore note from [3] that any four-dimensional Jordan-Osserman manifold has isotropic covariant derivative of the curvature, i.e., $\|\nabla R\| = 0$, although ∇R may be nonzero.

Remark 3.2. It follows from the work in [4] that any four-dimensional Osserman algebraic curvature tensor is Jordan-Osserman. However the existence of Osserman metrics which are not Jordan-Osserman was already pointed out in [11]. Indeed, note that the Jordan normal form of the Jacobi operators (2.5) corresponding to the metrics (2.1) changes from diagonalizable to nondiagonalizable according to (2.4). Moreover, since $24kf(x_4)f'(x_4)x_2 - 12kf''(x_4)x_1 + 3f(x_4)f''(x_4) + 4f'(x_4)^2$ defines a polynomial on x_1, x_2, it follows that any metric (2.1), when considered as globally defined in $\mathbb{R}^4$, changes its Jordan normal form, and thus, it is Osserman but not Jordan-Osserman. However, they restrict to Jordan-Osserman metrics on suitable open sets.

Remark 3.3. Since metrics (2.1) are not Jordan-Osserman in general, they are not curvature homogeneous, and thus they cannot be locally homogeneous. Moreover, even restricting to open sets where (2.1) defines a Jordan-Osserman metric (and hence 0-curvature homogeneous) they are not necessarily locally homogeneous. Indeed, for the special choice of $f(x_4) = x_4$, $(\mathbb{R}^4, g)$ is Jordan-Osserman in the open set $6kx_2x_4 \neq -1$. However, it is not locally homogeneous, since ∇R vanishes at any point $(0, 0, x_3, 0)$ and it is different from zero at those points $(0, 0, x_3, x_4)$ with $x_4 \neq 0$, which shows that it is not 1-curvature homogeneous.

Remark 3.4. Different kinds of Osserman manifolds may share the same eigenvalue structure. Indeed, the Jacobi operators of indefinite complex and paracomplex space forms have the same spectrum as that of metrics (2.1). Moreover, a straightforward calculation shows that metrics (2.1) have exactly the same second, fourth and sixth order scalar curvature invariants as the symmetric models. Recall that the main difference between complex and paracomplex space forms from the point of view of their Jacobi operators, is that the restriction of the metric to the subspace $E_{4k}(X) = span\{X\} \oplus ker(J_R(X) - 4kId)$ is definite in the complex case and indefinite in the paracomplex setting [6]. Moreover observe that any metric (2.1) induces a Lorentzian inner product on E_{4k}, since the Jacobi operators are nondiagonalizable. Indeed, it follows from the expression of the Jacobi operator associated to any non-null vector $X = \sum \alpha_i \partial_i$ that $-\alpha_4\partial_1 + \alpha_3\partial_2$ is a null eigenvector of $J_R(X)$ corresponding to the double eigenvalue k.

Remark 3.5. Let $Gr_k(T_pM)$ be the Grassmannian of nondegenerate k-planes in T_pM of a pseudo-Riemannian manifold (M, g). For each $E \in Gr_k(T_pM)$, let $J(E)$

denote the generalized Jacobi operator

$$J(E) = g(x_1, x_1)R(x_1, \cdot)x_1 + \cdots + g(x_k, x_k)R(x_k, \cdot)x_k,$$

where $\{x_1, \dots, x_k\}$ is an orthonormal basis for E. Then $J(E)$ is independent of the choice of orthonormal basis and (M, g) is said to be *k-Osserman at $p \in M$* if the eigenvalues of $J(E)$ counted with multiplicities are constant for every $E \in Gr_k(T_pM)$. Also, (M, g) is called *globally k-Osserman* if the characteristic polynomial of $J(E)$ is independent of $E \in \bigcup_{p\in M} Gr_k(T_pM)$. Further note that there is a certain kind of duality between the notions above since (M, g) is k-Osserman if and only if it is $(n-k)$-Osserman, where $n = \dim M$ [14].

Now, it follows from the work in [5] that a four-dimensional metric is 1-Osserman and 2-Osserman if and only if it is either of constant curvature or the Jacobi operators are two-step nilpotent. Therefore no metric (2.1) is Osserman of higher order (see [13], [14] for more information on higher order Osserman manifolds).

Remark 3.6. Finally, in order to give some motivation for metrics (2.1), recall that an specific feature of pseudo-Riemannian metrics is related to the local reductibility/decomposability of such structures [23]. It is a fact that many striking differences between the Riemannian and pseudo-Riemannian situations come from the existence of parallel degenerate distributions, which do not lead to local decompositions of the manifold. It was shown by Walker [22] that any four-dimensional metric equipped with a two-dimensional parallel degenerate distribution can be locally expressed in adapted coordinates $(x_1, \dots, x_4)$ by

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{3.1}$$

for arbitrary functions a, b, c depending on the variables $(x_1, \dots, x_4)$. Associated to any Walker metric (3.1) there is a natural almost para-Hermitian structure J (i.e., $J^2 = \mathrm{id}$, $g(J\cdot, J\cdot) = -g(\cdot, \cdot)$) defined by

$$J\partial_1 = -\partial_1, \quad J\partial_2 = \partial_2, \quad J\partial_3 = -a\partial_1 + \partial_3, \quad J\partial_4 = b\partial_2 - \partial_4,$$

which is integrable (i.e., the corresponding ± 1-eigenspaces define integrable distributions) if and only if $a_2 = 0$, $b_1 = 0$, where here and henceforth the subscript means partial derivative, i.e., $h_i = \frac{\partial}{\partial x_i} h$, for any function h depending on $(x_1, \dots, x_4)$ and $i = 1, \dots, 4$. Then, metrics (2.1) arise in the process of constructing para-Hermitian Einstein structures. Indeed, it can be shown from the Einstein equations that the para-Hermitian structure (g, J) is Einstein if and only if one of the following holds [8]
(i) The scalar curvature vanishes and the metric components are given by

$$\begin{aligned} a(x_1, x_2, x_3, x_4) &= x_1 P(x_3, x_4) + \gamma(x_3, x_4), \\ b(x_1, x_2, x_3, x_4) &= x_2 Q(x_3, x_4) + \delta(x_3, x_4), \\ c(x_1, x_2, x_3, x_4) &= x_1 \alpha(x_3, x_4) + x_2 \xi(x_3, x_4) + \eta(x_3, x_4), \end{aligned} \tag{3.2}$$

where P, Q, γ, δ, α, ξ and η are smooth functions satisfying

$$P\xi - \xi^2 + 2\xi_3 = 0, \quad Q\alpha - \alpha^2 + 2\alpha_4 = 0, \quad \alpha\xi + P_4 - \xi_4 + Q_3 - \alpha_3 = 0.$$

(ii) If the scalar curvature is nonzero, then the metric is given by

$$\begin{aligned} a(x_1,x_2,x_3,x_4) &= \tfrac{\kappa}{2}x_1^2 + x_1P(x_3,x_4) + \xi(x_3,x_4),\\ b(x_1,x_2,x_3,x_4) &= \tfrac{\kappa}{2}x_2^2 + x_2Q(x_3,x_4) + \eta(x_3,x_4),\\ c(x_1,x_2,x_3,x_4) &= \tfrac{1}{\kappa}\left(P_4(x_3,x_4) + Q_3(x_3,x_4)\right), \end{aligned} \tag{3.3}$$

for any smooth functions $P(x_3,x_4)$, $Q(x_3,x_4)$, $\xi(x_3,x_4)$, $\eta(x_3,x_4)$, or otherwise

$$\begin{aligned} a(x_1,x_2,x_3,x_4) &= \tfrac{\kappa}{3}x_1^2 + x_1P + \tfrac{3}{\kappa}\left(PS - S^2 + 2S_3\right),\\ b(x_1,x_2,x_3,x_4) &= \tfrac{\kappa}{3}x_2^2 + x_2Q + \tfrac{3}{\kappa}\left(QT - T^2 + 2T_4\right),\\ c(x_1,x_2,x_3,x_4) &= \tfrac{\kappa}{3}x_1x_2 + x_1T + x_2S + \tfrac{3}{\kappa}\left(ST + P_4 - T_3 + Q_3 - S_4\right), \end{aligned} \tag{3.4}$$

for any smooth functions $P(x_3,x_4)$, $S(x_3,x_4)$, $Q(x_3,x_4)$, $T(x_3,x_4)$.

Now it follows that metrics (3.3) cannot be Osserman, while those Ricci-flat metrics defined by (3.2) are Osserman with Jacobi operators either vanishing or nilpotent. Moreover, metrics (2.1) are obtained as a special case of (3.4).

References

[1] D. Alekseevsky, N. Blažić, V. Cortés, S. Vukmirović; A class of Osserman spaces, *J. Geom. Phys.* **53** (2005), 345–353.

[2] N. Blažić, N. Bokan, P. Gilkey; A note on Osserman Lorentzian manifolds, *Bull. London Math. Soc.* **29** (1997), 227–230.

[3] N. Blažić, N. Bokan, Z. Rakić; A note on the Osserman conjecture and isotropic covariant derivative of curvature, *Proc. Amer. Math. Soc.* **128** (1999), 245–253.

[4] N. Blažić, N. Bokan, Z. Rakić; Osserman pseudo-Riemannian manifolds of signature $(2,2)$, *J. Aust. Math. Soc.* **71** (2001), 367–395.

[5] A. Bonome, P. Castro, E. García-Río; Generalized Osserman four–dimensional manifolds, *Classical Quantum Gravity* **18** (2001), 4813–4822.

[6] A. Bonome, R. Castro, E. García-Río, L. Hervella, R. Vázquez-Lorenzo; Pseudo-Riemannian manifolds with simple Jacobi operators, *J. Math. Soc. Japan* **54** (2002), 847–875.

[7] Q. S. Chi; A curvature characterization of certain locally rank-one symmetric spaces, *J. Diff. Geom.* **28** (1988), 187–202.

[8] J. C. Díaz-Ramos, E. García-Río, R. Vázquez-Lorenzo; Osserman metrics on Walker 4-manifolds equipped with a para-Hermitian structure, to appear.

[9] E. García-Río, D. N. Kupeli, M. E. Vázquez-Abal; On a problem of Osserman in Lorentzian geometry, *Differential Geom. Appl.* **7** (1997), 85–100.

[10] E. García-Río, D. N. Kupeli, R. Vázquez-Lorenzo; *Osserman manifolds in semi-Riemannian geometry*, Lect. Notes Math. **1777**, Springer-Verlag, Berlin, Heidelberg, New York, 2002.

[11] E. García-Río, M. E. Vázquez-Abal, R. Vázquez-Lorenzo; Nonsymmetric Osserman pseudo-Riemannian manifolds, *Proc. Amer. Math. Soc.*, **126** (1998), 2771–2778.

[12] E. García-Río, R. Vázquez-Lorenzo; Four-dimensional Osserman symmetric spaces, *Geom. Dedicata* **88** (2001), 147–151.

[13] P. Gilkey; *Geometric Properties of Natural Operators Defined by the Riemannian Curvature Tensor*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[14] P. Gilkey, G. Stanilov, V. Videv; Pseudo-Riemannian manifolds whose generalized Jacobi operator has constant characteristic polynomial, *J. Geom.* **62** (1988), 144–153.

[15] P. Gilkey, R. Ivanova; Spacelike Jordan-Osserman algebraic curvature tensors in the higher signature setting, *Differential Geometry, Valencia, 2001*, 179–186, World Sci. Publ., River Edge, NJ, 2002.

[16] P. Gilkey, R. Ivanova; The Jordan normal form of Osserman algebraic curvature tensors, *Results Math.* **40** (2001), 192–204.

[17] P. Gilkey, R. Ivanova, I. Stavrov; Jordan Szabó algebraic covariant derivative curvature tensors, *Recent advances in Riemannian and Lorentzian geometries (Baltimore, MD, 2003)*, 65–75, Contemp. Math., **337**, Amer. Math. Soc., Providence, RI, 2003.

[18] P. Gilkey, I. Stavrov; Curvature tensors whose Jacobi or Szabó operator is nilpotent on null vectors, *Bull. London Math. Soc.* **34** (2002), 650–658.

[19] Y. Nikolayevsky; Osserman manifolds of dimension 8, *Manuscripta Math.* **115** (2004), 31–53.
[20] Y. Nikolayevsky; Osserman conjecture in dimension $\neq 8, 16$, *Math. Ann.* **331** (2005), 505–522.
[21] Z. I. Szabó; A short topological proof for the symmetry of 2 point homogeneous spaces, *Invent. Math.* **106** (1991), 61–64.
[22] A. G. Walker; Canonical form for a Riemannian space with a parallel field of null planes, *Quart. J. Math. Oxford* **(2) 1** (1950), 69–79.
[23] H. Wu; On the de Rham decomposition theorem, *Illinois J. Math.* **8** (1964), 291-311.

Department of Geometry and Topology, Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain
E-mail address: xtjosec@usc.es xtedugr@usc.es ravazlor@usc.es